THE GROMOV-WITTEN AND DONALDSON-THOMAS CORRESPONDENCE FOR TRIVIAL ELLIPTIC FIBRATIONS

DAN EDIDIN[1] AND ZHENBO QIN[2]

Abstract. We study the Gromov-Witten and Donaldson-Thomas correspondence conjectured in [MNOP1, MNOP2] for trivial elliptic fibrations. In particular, we verify the Gromov-Witten and Donaldson-Thomas correspondence for primary fields when the threefold is $E \times S$ where E is a smooth elliptic curve and S is a smooth surface with numerically trivial canonical class.

1. Introduction

The correspondence between the Gromov-Witten theory and Donaldson-Thomas theory for threefolds was conjectured and studied in [MNOP1, MNOP2]. Since then, it has been investigated extensively (see [MP, JLi, Kat, KLQ, Beh, BF2] and the references there). A relationship between the quantum cohomology of the Hilbert scheme of points in the complex plane and the Gromov-Witten and Donaldson-Thomas correspondence for local curves was proved in [OP2, OP3]. The equivariant version was proposed and partially verified in [BP, GS]. In this paper, we study the Gromov-Witten and Donaldson-Thomas correspondence when the threefold admits a trivial elliptic fibration.

To state our results, we introduce some notation and refer to Subsect. 2.1 and Subsect. 3.1 for details. Let X be a complex threefold, $\gamma_1, \ldots, \gamma_r \in H^*(X; \mathbb{Q})$,

$$\beta \in H_2(X; \mathbb{Z}) \backslash \{0\},$$

$k_1, \ldots, k_r$ be nonnegative integers, and u, q be formal variables. Let

$$\mathbf{Z}'_{\rm GW}\left(X; u \Big| \prod_{i=1}^r \tau_{k_i}(\gamma_i)\right)_\beta, \qquad \mathbf{Z}'_{\rm DT}\left(X; q \Big| \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i)\right)_\beta$$

be the reduced degree-β partition functions for the descendent Gromov-Witten invariants and Donaldson-Thomas invariants of X respectively.

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Conjecture 1.1. ([MNOP1, MNOP2]) Let $\beta \in H_2(X;\mathbb{Z})\backslash\{0\}$ and $\mathfrak{d} = -\int_\beta K_X$. Then after the change of variables $e^{iu} = -q$, we have

$$(-iu)^{\mathfrak{d}}\,\mathbf{Z}'_{\mathrm{GW}}\left(X;u|\prod_{i=1}^r \tau_0(\gamma_i)\right)_\beta = (-q)^{-\mathfrak{d}/2}\,\mathbf{Z}'_{\mathrm{DT}}\left(X;q|\prod_{i=1}^r \tilde{\tau}_0(\gamma_i)\right)_\beta. \tag{1.1}$$

Theorem 1.2. *Let $f: X = E \times S \to S$ be the projection where E is an elliptic curve and S is a smooth surface. Then the Gromov-Witten/Donaldson-Thomas correspondence (1.1) holds if either $\int_\beta K_X = \int_\beta f^*K_S = 0$, or*

$$\gamma_1, \ldots, \gamma_r \in f^*H^*(S;\mathbb{Q}) \subset H^*(X;\mathbb{Q}).$$

Proof. The conclusion follows from Proposition 2.6 and Proposition 3.6 when

$$\int_\beta K_X = \int_\beta f^*K_S = 0.$$

It follows from Proposition 2.7 and Proposition 3.7 when

$$\gamma_1, \ldots, \gamma_r \in f^*H^*(S;\mathbb{Q}) \subset H^*(X;\mathbb{Q}).$$

□

Corollary 1.3. *Let E be an elliptic curve and S be a smooth surface with numerically trivial canonical class K_S. Then the Gromov-Witten/Donaldson-Thomas correspondence (1.1) holds for the threefold $X = E \times S$.*

In fact, when $\gamma_1, \ldots, \gamma_r \in f^*H^*(S;\mathbb{Q}) \subset H^*(X;\mathbb{Q})$, Proposition 2.7 and Proposition 3.7 state that after the change of variables $e^{iu} = -q$,

$$(-iu)^{\mathfrak{d}-\sum_i k_i}\,\mathbf{Z}'_{\mathrm{GW}}\left(X;u|\prod_{i=1}^r \tau_{k_i}(\gamma_i)\right)_\beta = (-q)^{-\mathfrak{d}/2}\,\mathbf{Z}'_{\mathrm{DT}}\left(X;q|\prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i)\right)_\beta.$$

This is consistent with (and partially sharpens) the Conjecture 4 in [MNOP2] which is about the Gromov-Witten and Donaldson-Thomas correspondence for descendent fields. It would be interesting to see whether this sharpened version holds for general cohomology classes $\gamma_1, \ldots, \gamma_r \in H^*(X;\mathbb{Q})$.

Proposition 2.7 and Proposition 3.7 are proved in Sect. 2 and Sect. 3 respectively. The idea is to view the elliptic curve E as an algebraic group and to use the action of E on the moduli space $\overline{\mathfrak{M}}_{g,r}(X,\beta)$ of stable maps and the moduli space $\mathfrak{I}_n(X,\beta)$ of ideal sheaves. The E-action on $\overline{\mathfrak{M}}_{g,r}(X,\beta)$ has no fixed points when $r \geq 1$, or $g \neq 1$, or $\beta \neq d\beta_0$. It follows from Lemma 2.5 that the corresponding Gromov-Witten invariants are zero. The only exception is $\langle\rangle_{1,d\beta_0}$ which can be computed directly by using the work of Okounkov-Pandharipande [OP1] on the Gromov-Witten invariants of an elliptic curve and Göttsche's formula for the Euler characteristics of the Hilbert scheme $S^{[d]}$ of points on a smooth surface S. Similarly, the E-action on $\mathfrak{I}_n(X,\beta)$ has no fixed points when $n \geq 1$ or $\beta \neq d\beta_0$. It follows from Lemma 3.5 that the corresponding Donaldson-Thomas invariants are also zero. The only exception is $\langle\rangle_{0,d\beta_0}$ which can be computed directly by determining the obstruction bundle over the moduli space $\mathfrak{I}_0(X,d\beta_0) \cong S^{[d]}$.

It is expected that our approach can be used to handle the *relative* Gromov-Witten and Donaldson-Thomas correspondence (see [MNOP2]) for trivial elliptic fibrations. In another direction, one might attempt to study the (absolute and relative) Gromov-Witten and Donaldson-Thomas correspondence for nontrivial elliptic fibrations. We leave these to the interested readers.

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2. Gromov-Witten theory

2.1. Gromov-Witten invariants.

Let X be a smooth projective complex variety. Fix $\beta \in H_2(X;\mathbb{Z})$. Let $\overline{\mathfrak{M}}_{g,r}(X,\beta)$ be the moduli space of stable maps from connected genus-g curves with r marked points to X representing the class β. The virtual fundamental class $[\overline{\mathfrak{M}}_{g,r}(X,\beta)]^{\rm vir}$ has been constructed in [BF1, LT]. By ignoring the extra notation of stacks, the virtual fundamental class $[\overline{\mathfrak{M}}_{g,r}(X,\beta)]^{\rm vir}$ is defined by the element

$$R(\pi_{g,r})_*(\mathrm{ev}_{r+1})^*T_X \tag{2.1}$$

in the derived category $\mathfrak{D}_{\rm coh}(\overline{\mathfrak{M}}_{g,r}(X,\beta))$ of coherent sheaves on $\overline{\mathfrak{M}}_{g,r}(X,\beta)$, where

$$\mathrm{ev}_i:\ \overline{\mathfrak{M}}_{g,r+1}(X,\beta)\to X$$

is the i-th evaluation map, and $\pi_{g,r}$ stands for the morphism:

$$\pi_{g,r}:\ \overline{\mathfrak{M}}_{g,r+1}(X,\beta)\to \overline{\mathfrak{M}}_{g,r}(X,\beta) \tag{2.2}$$

forgetting the $(r+1)$-th marked point. Let $\mathcal{L}_i$ be the cotangent line bundle on $\overline{\mathfrak{M}}_{g,r}(X,\beta)$ associated to the i-th marked point. Put

$$\psi_i = c_1(\mathcal{L}_i).$$

For $\gamma_1,\ldots,\gamma_r \in H^*(X;\mathbb{Q})$ and nonnegative integers $k_1,\ldots,k_r$, define

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)\rangle_{g,\beta} = \int_{[\overline{\mathfrak{M}}_{g,r}(X,\beta)]^{\rm vir}} \prod_{i=1}^r \psi_i^{k_i}\mathrm{ev}_i^*(\gamma_i). \tag{2.3}$$

Define the *reduced* Gromov-Witten potential of X by

$$\mathbf{F}'_{\rm GW}\left(X;u,v\Big|\prod_{i=1}^r \tau_{k_i}(\gamma_i)\right) = \sum_{\beta\neq 0}\sum_{g\geq 0}\left\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i)\right\rangle_{g,\beta} u^{2g-2}v^\beta \tag{2.4}$$

omitting the constant maps. For $\beta\neq 0$, the *reduced partition function*

$$\mathbf{Z}'_{\rm GW}\left(X;u\Big|\prod_{i=1}^r \tau_{k_i}(\gamma_i)\right)_\beta$$

of degree-β Gromov-Witten invariants is defined by setting:

$$1+\sum_{\beta\neq 0}\mathbf{Z}'_{\rm GW}\left(X;u\Big|\prod_{i=1}^r \tau_{k_i}(\gamma_i)\right)_\beta v^\beta \;=\; \exp\,\mathbf{F}'_{\rm GW}\left(X;u,v\Big|\prod_{i=1}^r \tau_{k_i}(\gamma_i)\right). \tag{2.5}$$

Alternatively, let $\overline{\mathfrak{M}}'_{g,r}(X,\beta)$ be the moduli space of stable maps from *possibly disconnected* curves C of genus-g with r marked points and with no collapsed connected components. Here the genus of a possibly disconnected curve C is

$$1-\chi(\mathcal{O}_C) = 1-\ell+\sum_{i=1}^{\ell} g_{C_i}$$

where $C_1,\ldots,C_\ell$ denote all the connected components of C. For $\gamma_1,\ldots,\gamma_r \in H^*(X;\mathbb{Q})$ and $k_1,\ldots,k_r \ge 0$, define the reduced Gromov-Witten invariant by

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_r}(\gamma_r)\rangle'_{g,\beta} = \int_{[\overline{\mathfrak{M}}'_{g,r}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{r} \psi_i^{k_i}\mathrm{ev}_i^*(\gamma_i). \tag{2.6}$$

Then the reduced partition function of degree-β invariants is also given by

$$\mathbf{Z}'_{\mathrm{GW}}\left(X;u\Big|\prod_{i=1}^{r}\tau_{k_i}(\gamma_i)\right)_\beta = \sum_{g\in\mathbb{Z}}\left\langle\prod_{i=1}^{r}\tau_{k_i}(\gamma_i)\right\rangle'_{g,\beta} u^{2g-2}. \tag{2.7}$$

When $\dim(X)=3$, the expected dimensions of $\overline{\mathfrak{M}}_{g,r}(X,\beta)$ and $\overline{\mathfrak{M}}'_{g,r}(X,\beta)$ are

$$-\int_\beta K_X + r. \tag{2.8}$$

Remark 2.1. By the Fundamental Class Axiom, Divisor Axiom and Dilation Axiom of the descendent Gromov-Witten invariants, if $\beta \ne 0$ and $\displaystyle\int_\beta K_X = 0$, then

$$\mathbf{Z}'_{\mathrm{GW}}\left(X;u\Big|\prod_{i=1}^{r}\tau_{k_i}(\gamma_i)\right)_\beta$$

can be reduced to the case $r=0$, i.e., to the reduced partition function

$$\mathbf{Z}'_{\mathrm{GW}}(X;u)_\beta. \tag{2.9}$$

2.2. **The computations.**

We begin with the Gromov-Witten invariants of a smooth elliptic curve E. Let $d\ge 1$ and $[E]\in H_2(E;\mathbb{Z})$ be the fundamental class. We use

$$\overline{\mathfrak{M}}_{g,r}(E,d),\ \ \overline{\mathfrak{M}}'_{g,r}(E,d)$$

to denote the moduli spaces $\overline{\mathfrak{M}}_{g,r}(E,d[E])$, $\overline{\mathfrak{M}}'_{g,r}(E,d[E])$ respectively. The expected dimension of the moduli spaces $\overline{\mathfrak{M}}_{1,0}(E,d)$ and $\overline{\mathfrak{M}}'_{1,0}(E,d)$ is zero. So

$$\langle\rangle_{1,d[E]} = \deg\big[\overline{\mathfrak{M}}_{1,0}(E,d)\big]^{\mathrm{vir}}, \tag{2.10}$$

$$\langle\rangle'_{1,d[E]} = \deg\big[\overline{\mathfrak{M}}'_{1,0}(E,d)\big]^{\mathrm{vir}}. \tag{2.11}$$

Note that if C is the (possibly disconnected) domain curve of a stable map in $\overline{\mathfrak{M}}'_{1,0}(E,d)$, then every connected component of C must be of genus-1. Therefore,

as in (2.4), (2.5) and (2.7), we obtain the following relation:

$$1+\sum_{d=1}^{+\infty}\langle\rangle'_{1,d[E]}\ v^d \ = \ \exp\ \sum_{d=1}^{+\infty}\langle\rangle_{1,d[E]}\ v^d. \tag{2.12}$$

By the Theorem 5 in [OP1] (replacing n and q there by 0 and v respectively),

$$1+\sum_{d=1}^{+\infty}\langle\rangle'_{1,d[E]}\ v^d \ = \ \frac{1}{\prod_{m=0}^{+\infty}(1-v^m)}. \tag{2.13}$$

In the rest of this section, we adopt the following notation.

Notation 2.2. (i) Let $X = E \times S$ where E is an elliptic curve and S is a smooth surface. Let $\beta_0 \in H_2(X;\mathbb{Z})$ be the fiber class of the fibration

$$f : X = E \times S \to S.$$

We use K_X to denote both the canonical class and the canonical line bundle of X.

(ii) For $d \geq 0$, let $S^{[d]}$ be the Hilbert scheme which parametrizes the length-d 0-dimensional closed subschemes of the surface S.

(iii) Fix $O \in E$ as the zero element for the group law on E. For $p \in E$, let

$$\phi_p : E \to E \tag{2.14}$$

be the automorphism of E defined via translation $\phi_p(e) = p+e$. We have an action of E on $X = E \times S$ via the automorphisms $\phi_p \times \mathrm{Id}_S,\ p \in E$.

Lemma 2.3. *Let X be from Notation 2.2 and $d \geq 1$. Then, we have*

$$\langle\rangle'_{1,d\beta_0} = \chi\big(S^{[d]}\big).$$

Proof. First of all, let $\mathcal{H}_1^E$ be the rank-1 Hodge bundle over $\overline{\mathfrak{M}}_{1,0}(E,d)$, i.e.,

$$\mathcal{H}_1^E = (\pi_{1,0})_*\omega_{1,0}$$

where $\omega_{1,0}$ is the relative dualizing sheaf of the forgetful map $\pi_{1,0}$ in (2.2).

Next, by the universal property of moduli spaces, we have

$$\overline{\mathfrak{M}}_{1,0}(X, d\beta_0) \cong \overline{\mathfrak{M}}_{1,0}(E,d) \times S. \tag{2.15}$$

By the definitions of virtual fundamental classes and the Hodge bundle,

$$\big[\overline{\mathfrak{M}}_{1,0}(X,d\beta_0)\big]^{\mathrm{vir}} = e\left(\pi_1^*\big(\mathcal{H}_1^E\big)^{\mathrm{v}} \otimes \pi_2^* T_S\right) \cap \pi_1^*\big[\overline{\mathfrak{M}}_{1,0}(E,d)\big]^{\mathrm{vir}} \tag{2.16}$$

where π_1 and π_2 are the two projections of $\overline{\mathfrak{M}}_{1,0}(X,d\beta_0)$ via the isomorphism (2.15), and $e(\cdot)$ denotes the Euler class (or the top class). Note that

$$e\big(\pi_1^*\big(\mathcal{H}_1^E\big)^{\mathrm{v}} \otimes \pi_2^*T_S\big) = \pi_2^*e(S) + \pi_2^*K_S \cdot \pi_1^*c_1\big(\mathcal{H}_1^E\big) + \pi_1^*c_1\big(\mathcal{H}_1^E\big)^2.$$

By (2.16), $\langle\rangle_{1,d\beta_0} = \chi(S)\cdot\langle\rangle_{1,d[E]}$. Therefore, we obtain

$$\begin{aligned} 1+\sum_{d=1}^{+\infty}\langle\rangle'_{1,d\beta_0}\, v^d &= \exp\sum_{d=1}^{+\infty}\langle\rangle_{1,d\beta_0}\, v^d \\ &= \exp\left(\chi(S)\cdot\sum_{d=1}^{+\infty}\langle\rangle_{1,d[E]}\, v^d\right) \\ &= \left(1+\sum_{d=1}^{+\infty}\langle\rangle'_{1,d[E]}\, v^d\right)^{\chi(S)} \\ &= \frac{1}{\prod_{m=0}^{+\infty}(1-v^m)^{\chi(S)}} \end{aligned} \tag{2.17}$$

by (2.12) and (2.13). By Göttsche's formula in [Got] for $\chi\big(S^{[d]}\big)$, we have

$$\sum_{d=0}^{+\infty}\chi\big(S^{[d]}\big)\, v^d = \frac{1}{\prod_{m=0}^{+\infty}(1-v^m)^{\chi(S)}}.$$

Combining this with (2.17), we conclude that $\langle\rangle'_{1,d\beta_0} = \chi\big(S^{[d]}\big)$. □

Let X be from Notation 2.2 and $\beta \in H_2(X;\mathbb{Z})\backslash\{0\}$. For any $p \in E$,

$$(\phi_p \times \mathrm{Id}_S)_*\beta = \beta \tag{2.18}$$

since $\{\phi_p \times \mathrm{Id}_S\}_{p\in E}$ form a connected algebraic family of automorphims of X. Thus the algebraic group E acts on the stack of r-pointed degree-β stable maps to X (see [Kon]). The universal properties of moduli spaces imply that there is a corresponding action of E on the moduli space $\overline{\mathfrak{M}}_{g,r}(X,\beta)$. For $p \in E$, let

$$\Psi_p : \overline{\mathfrak{M}}_{g,r}(X,\beta) \to \overline{\mathfrak{M}}_{g,r}(X,\beta)$$

be the corresponding automorphism. Then we see that the automorphism Ψ_p maps a point $[\mu : (C; w_1,\ldots,w_r) \to X] \in \overline{\mathfrak{M}}_{g,r}(X,\beta)$ to the point

$$[(\phi_p \times \mathrm{Id}_S)\circ\mu : (C; w_1,\ldots,w_r) \to X] \in \overline{\mathfrak{M}}_{g,r}(X,\beta). \tag{2.19}$$

Lemma 2.4. *With the notation as above, the algebraic group E acts without fixed points on $\overline{\mathfrak{M}}_{g,r}(X,\beta)$ if $\beta \ne d\beta_0$, or $r \ge 1$, or $g \ne 1$.*

Proof. Assume that $[\mu : (C; w_1,\ldots,w_r) \to X] \in \overline{\mathfrak{M}}_{g,r}(X,\beta)$ is fixed by the action of E. By definition, for every $p \in E$, there is an automorphism τ_p of C such that

$$\mu\circ\tau_p = (\phi_p \times \mathrm{Id}_S)\circ\mu \tag{2.20}$$

and $\tau_p(w_i) = w_i$ for all $1 \le i \le r$. In particular, for every $p \in E$, we have

$$\mu(C) = (\phi_p \times \mathrm{Id}_S)\big(\mu(C)\big).$$

So $\mu(C)$ is a fiber of the elliptic fibration f, and $\beta = d\beta_0$ for some $d \ge 1$. By our assumption, either $r \ge 1$ or $g \ge 2$. By (2.20), we get

$$\mu\circ\tau_p(C) = \phi_p\big(\mu(C)\big). \tag{2.21}$$

Since ϕ_p acts freely on the fiber $\mu(C)$, (2.21) implies that the automorphisms τ_p of the marked curve $(C; w_1, \ldots, w_r)$ are different for different points $p \in E$. Hence the automorphism group of the marked curve $(C; w_1, \ldots, w_r)$ is infinite. This is impossible since either $g \geq 2$ or $g = 1$ and $r \geq 1$. □

Lemma 2.5. *Let $\beta \in H_2(X;\mathbb{Z})\backslash\{0\}$. Assume that $\gamma_1, \ldots, \gamma_r \in f^*H^*(S;\mathbb{Q}) \subset H^*(X;\mathbb{Q})$. If $\beta \neq d\beta_0$, or $r \geq 1$, or $g \neq 1$, then we have*

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle'_{g,\beta} = 0.$$

Proof. First of all, note that it suffices to show that

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle_{g,\beta} = 0 \tag{2.22}$$

if $\beta \neq d\beta_0$, or $r \geq 1$, or $g \neq 1$. In the following, we prove (2.22).

By Lemma 2.4, E acts without fixed points on $\overline{\mathfrak{M}}_{g,r}(X,\beta)$. Since E is an elliptic curve, any proper algebraic subgroup is finite. Thus the stabilizer of any point for the E action on $\overline{\mathfrak{M}}_{g,r}(X,\beta)$ is finite. Since $\overline{\mathfrak{M}}_{g,r}(X,\beta)$ is finite type, the order of the stabilizer subgroup at any point is bounded by some number N. Thus, if G is a cyclic subgroup of E of prime order $p > N$, then G acts freely on $\overline{\mathfrak{M}}_{g,r}(X,\beta)$. We fix such a cyclic subgroup G of E in the rest of the proof.

The complex $R(\pi_{g,r})_*(\mathrm{ev}_{r+1})^*T_X$ from (2.1) is equivariant for the action of any algebraic automorphism group of X. Thus for some positive integer m (independent of G), the cycle $m\big[\overline{\mathfrak{M}}_{g,r}(X,\beta)\big]^{\mathrm{vir}}$ defines an element of the integral equivariant Borel-Moore homology group $H^G_*\big(\overline{\mathfrak{M}}_{g,r}(X,\beta)\big)$. Likewise if $\gamma_i \in f^*H^*(S;\mathbb{Q})$, then the cycle γ_i is invariant under the action of E on X. Hence some positive multiple $m_i\gamma_i$ defines an element of $H^*_G(X)$, where m_i is independent of G. Note from (2.19) that the evaluation map $\mathrm{ev}_i : \overline{\mathfrak{M}}_{g,r}(X,\beta) \to X$ is G-equivariant, so the pullback $\mathrm{ev}_i^*(m_i\gamma_i)$ determines an element of $H^*_G(\overline{\mathfrak{M}}_{g,r}(X,\beta))$. In addition, the cotangent line bundles $\mathcal{L}_i$ $(1 \leq i \leq r)$ over $\overline{\mathfrak{M}}_{g,r}(X,\beta)$ are equivariant for the action of G. It follows from the definition (2.3) that the cycle

$$mm_1 \cdots m_r \, \langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle_{g,\beta}$$

defines an element in the degree-0 Borel-Moore homology $H^G_0\big(\overline{\mathfrak{M}}_{g,r}(X,\beta)\big)$.

Since G is a cyclic subgroup of order p which acts freely on $\overline{\mathfrak{M}}_{g,r}(X,\beta)$, any element of $H^G_0\big(\overline{\mathfrak{M}}_{g,r}(X,\beta)\big)$ is represented by a G-invariant 0-cycle whose degree is a multiple of p (possibly 0). Since p can be taken to be arbitrarily large,

$$mm_1 \cdots m_r \, \langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle_{g,\beta} = 0.$$

Therefore, $\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle_{g,\beta} = 0$. This completes the proof of (2.22). □

We define the cohomology degree $|\gamma| = \ell$ when $\gamma \in H^\ell(X;\mathbb{Q})$.

Proposition 2.6. *Let $\beta \in H_2(X;\mathbb{Z})\backslash\{0\}$. Assume $\int_\beta K_X = \int_\beta f^*K_S = 0$. Then,*

$$\mathbf{Z}'_{\mathrm{GW}}\left(X; u \Big| \prod_{i=1}^{r} \tau_0(\gamma_i)\right)_\beta$$
$$= \begin{cases} \prod_{i=1}^r \int_\beta \gamma_i \cdot \chi\big(S^{[d]}\big) & \text{if } |\gamma_i| = 2 \text{ for every } i \text{ and } \beta = d\beta_0 \text{ for some } d \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By (2.8) and the degree condition on Gromov-Witten invariants,

$$\sum_{i=1}^r |\gamma_i| = 2r.$$

By the Fundamental Class Axiom and Divisor Axiom of Gromov-Witten invariants,

$$\langle \tau_0(\gamma_1) \cdots \tau_0(\gamma_r) \rangle_{g,\beta} = \begin{cases} \prod_{i=1}^r \int_\beta \gamma_i \cdot \langle \rangle_{g,\beta} & \text{if } |\gamma_i| = 2 \text{ for every } i; \\ 0 & \text{otherwise.} \end{cases}$$

So by Lemma 2.3 and by taking $r = 0$ in (2.22), we conclude that

$$\langle \tau_0(\gamma_1) \cdots \tau_0(\gamma_r) \rangle'_{g,\beta}$$
$$= \begin{cases} \prod_{i=1}^r \int_\beta \gamma_i \cdot \chi\big(S^{[d]}\big) & \text{if } |\gamma_i| = 2 \text{ for every } i, g = 1, \beta = d\beta_0; \\ 0 & \text{otherwise.} \end{cases}$$

Now our proposition follows directly from the identity (2.7). ☐

Proposition 2.7. *Let X be from Notation 2.2 and $\beta \in H_2(X;\mathbb{Z})\backslash\{0\}$. Assume that $\gamma_1, \ldots, \gamma_r \in f^*H^*(S;\mathbb{Q}) \subset H^*(X;\mathbb{Q})$. Then,*

$$\mathbf{Z}'_{\mathrm{GW}}\left(X; u \Big| \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\right)_\beta = \begin{cases} \chi\big(S^{[d]}\big) & \text{if } r = 0 \text{ and } \beta = d\beta_0 \text{ with } d \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Follows from the identity (2.7), Lemma 2.3 and Lemma 2.5. ☐

3. Donaldson-Thomas theory

3.1. Donaldson-Thomas invariants.

Let X be a smooth projective complex threefold. For a fixed class $\beta \in H_2(X;\mathbb{Z})$ and a fixed integer n, following the definition and notation in [MNOP1, MNOP2], we define $\mathfrak{I}_n(X,\beta)$ to be the moduli space parametrizing the ideal sheaves I_Z of 1-dimensional closed subschemes Z of X satisfying the conditions:

$$\chi(\mathcal{O}_Z) = n, \qquad [Z] = \beta \tag{3.1}$$

where $[Z]$ is the class associated to the dimension-1 component (weighted by their intrinsic multiplicities) of Z. Note that $\mathfrak{I}_n(X,\beta)$ is a special case of the moduli spaces of Gieseker semistable torsion-free sheaves over X. When the anti-canonical divisor $-K_X$ is effective, perfect obstruction theories on the moduli spaces $\mathfrak{I}_n(X,\beta)$

have been constructed in [Tho]. This result has been generalized in [MP]. By the Lemma 1 in [MNOP2], the virtual dimension of $\mathfrak{I}_n(X,\beta)$ is

$$-\int_\beta K_X. \tag{3.2}$$

The Donaldson-Thomas invariant is defined via integration against the virtual fundamental class $[\mathfrak{I}_n(X,\beta)]^{\mathrm{vir}}$ of the moduli space $\mathfrak{I}_n(X,\beta)$. More precisely, let $\gamma \in H^\ell(X;\mathbb{Q})$ and $\mathcal{I}$ be the universal ideal sheaf over $\mathfrak{I}_n(X,\beta)\times X$. Let

$$\mathfrak{ch}_{k+2}(\gamma):\ H_*\big(\mathfrak{I}_n(X,\beta);\mathbb{Q}\big) \to H_{*-2k+2-\ell}\big(\mathfrak{I}_n(X,\beta);\mathbb{Q}\big) \tag{3.3}$$

be the operation on the homology of $\mathfrak{I}_n(X,\beta)$ defined by

$$\mathfrak{ch}_{k+2}(\gamma)(\xi) = \pi_{1*}\big(\mathrm{ch}_{k+2}(\mathcal{I})\cdot \pi_2^*\gamma \cap \pi_1^*\xi\big) \tag{3.4}$$

where π_1 and π_2 be the two projections on $\mathfrak{I}_n(X,\beta)\times X$. Define

$$\begin{aligned} & \langle \tilde\tau_{k_1}(\gamma_1)\cdots\tilde\tau_{k_r}(\gamma_r)\rangle_{n,\beta} \\ = & \int_{[\mathfrak{I}_n(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^r (-1)^{k_i+1}\mathfrak{ch}_{k_i+2}(\gamma_i) \\ = & (-1)^{k_1+1}\mathfrak{ch}_{k_1+2}(\gamma_1)\circ\cdots\circ(-1)^{k_r+1}\mathfrak{ch}_{k_r+2}(\gamma_r)\big([\mathfrak{I}_n(X,\beta)]^{\mathrm{vir}}\big). \end{aligned} \tag{3.5}$$

The partition function for these descendent Donaldson-Thomas invariants is

$$\mathbf{Z}_{\mathrm{DT}}\left(X;q\mid \prod_{i=1}^r \tilde\tau_{k_i}(\gamma_i)\right)_\beta = \sum_{n\in\mathbb{Z}} \langle \tilde\tau_{k_1}(\gamma_1)\cdots\tilde\tau_{k_r}(\gamma_r)\rangle_{n,\beta}\ q^n. \tag{3.6}$$

The partition function for the degree-0 Donaldson-Thomas invariants of X is

$$\mathbf{Z}_{\mathrm{DT}}(X;q)_0 = M(-q)^{\chi(X)} \tag{3.7}$$

by [JLi, BF2] (this formula was conjectured in [MNOP1, MNOP2]), where

$$M(q) = \prod_{n=1}^{+\infty}\frac{1}{(1-q^n)^n}$$

is the McMahon function. The *reduced partition function* is defined to be

$$\begin{aligned} \mathbf{Z}'_{\mathrm{DT}}\left(X;q\mid \prod_{i=1}^r \tilde\tau_{k_i}(\gamma_i)\right)_\beta &= \frac{\mathbf{Z}_{\mathrm{DT}}\left(X;q\mid \prod_{i=1}^r \tilde\tau_{k_i}(\gamma_i)\right)_\beta}{\mathbf{Z}_{\mathrm{DT}}(X,q)_0} \\ &= \frac{\mathbf{Z}_{\mathrm{DT}}\left(X;q\mid \prod_{i=1}^r \tilde\tau_{k_i}(\gamma_i)\right)_\beta}{M(-q)^{\chi(X)}}. \end{aligned} \tag{3.8}$$

In the next two lemmas, we study the operators $\mathfrak{ch}_2(\gamma)$ and $\mathfrak{ch}_3(1_X)$ respectively, where $1_X \in H^*(X;\mathbb{Q})$ is the fundamental cohomology class. The results will be used in Subsect. 3.2. Note that the first lemma is the analogue to the Fundamental Class Axiom and Divisor Axiom of Gromov-Witten invariants, while the second one is the analogue to the Dilaton Axiom of Gromov-Witten invariants. By (3.3),

$$\mathfrak{ch}_2(\gamma):\ H_b\big(\mathfrak{I}_n(X,\beta);\mathbb{Q}\big)\to H_{b-2+|\gamma|}\big(\mathfrak{I}_n(X,\beta);\mathbb{Q}\big),$$

$$\mathfrak{ch}_3(1_X): \ H_b\big(\mathfrak{I}_n(X,\beta);\mathbb{Q}\big) \to H_b\big(\mathfrak{I}_n(X,\beta);\mathbb{Q}\big).$$

Let $cl : A_*\big(\mathfrak{I}_n(X,\beta)\big) \otimes \mathbb{Q} \to H_*\big(\mathfrak{I}_n(X,\beta);\mathbb{Q}\big)$ be the cycle map. Put

$$H_*^{\rm alg}\big(\mathfrak{I}_n(X,\beta)\big) = {\rm im}(cl).$$

Lemma 3.1. (i) *Let $\beta \in H_2(X;\mathbb{Z})$ and $\gamma \in H^\ell(X;\mathbb{Q})$. Then,*

$$\mathfrak{ch}_2(\gamma)|_{H_*^{\rm alg}(\mathfrak{I}_n(X,\beta))} = \begin{cases} 0 & \text{if } \ell = 0 \text{ or } 1; \\ -\int_\beta \gamma \cdot {\rm Id} & \text{if } \ell = 2. \end{cases}$$

(ii) *If the moduli space $\mathfrak{I}_n(X,\beta)$ is smooth, then*

$$\mathfrak{ch}_2(\gamma) = \begin{cases} 0 & \text{if } \ell = 0 \text{ or } 1; \\ -\int_\beta \gamma \cdot {\rm Id} & \text{if } \ell = 2. \end{cases}$$

Proof. (i) Let $\mathfrak{I} = \mathfrak{I}_n(X,\beta)$. By [FG], there is a proper morphism

$$p: \ \widetilde{\mathfrak{I}} \to \mathfrak{I}$$

with $\widetilde{\mathfrak{I}}$ smooth and $p_*: H_*^{\rm alg}(\widetilde{\mathfrak{I}}) \to H_*^{\rm alg}(\mathfrak{I})$ surjective. Such a morphism p is called a *nonsingular envelope* (see p.299 of [FG]). Let $\tilde\pi_1$ and $\tilde\pi_2$ be the projections from $\widetilde{\mathfrak{I}} \times X$ to the first and second factors respectively.

Let $\xi \in H_*^{\rm alg}(\mathfrak{I})$. Then $\xi = p_*\tilde\xi$ for some $\tilde\xi \in H_*^{\rm alg}(\widetilde{\mathfrak{I}})$. Define

$$\tilde{\mathfrak{ch}}_2(\gamma)(\tilde\xi) = \tilde\pi_{1*}\left({\rm ch}_2\big((p\times {\rm Id}_X)^*\mathcal{I}\big)\,\tilde\pi_2^*\gamma \cap \tilde\pi_1^*\tilde\xi\right) \tag{3.9}$$

where $\mathcal{I}$ denotes the universal ideal sheaf over $\mathfrak{I}\times X$. Using the projection formula and the fact that $(p\times {\rm Id}_X)_*\tilde\pi_1^*\tilde\xi = \pi_1^*p_*\tilde\xi = \pi_1^*\xi$, we have

$$\begin{aligned} p_*\big(\tilde{\mathfrak{ch}}_2(\gamma)(\tilde\xi)\big) &= p_*\tilde\pi_{1*}\left({\rm ch}_2\big((p\times {\rm Id}_X)^*\mathcal{I}\big)\,\tilde\pi_2^*\gamma \cap \tilde\pi_1^*\tilde\xi\right) \\ &= \pi_{1*}(p\times {\rm Id}_X)_*\left((p\times {\rm Id}_X)^*\big({\rm ch}_2(\mathcal{I})\,\pi_2^*\gamma\big) \cap \tilde\pi_1^*\tilde\xi\right) \\ &= \pi_{1*}\left({\rm ch}_2(\mathcal{I})\,\pi_2^*\gamma \cap (p\times {\rm Id}_X)_*\tilde\pi_1^*\tilde\xi\right) \\ &= \mathfrak{ch}_2(\gamma)(\xi). \end{aligned} \tag{3.10}$$

Since $\widetilde{\mathfrak{I}}$ is smooth, the Poincaré duality holds and we see from (3.9) that

$$\tilde{\mathfrak{ch}}_2(\gamma)(\tilde\xi) = \tilde\pi_{1*}\big({\rm ch}_2((p\times {\rm Id}_X)^*\mathcal{I})\tilde\pi_2^*\gamma\big) \cap \tilde\xi$$

where $\tilde\pi_{1*}\big({\rm ch}_2((p\times {\rm Id}_X)^*\mathcal{I})\tilde\pi_2^*\gamma\big)$ is the cohomology class Poincaré dual to

$$\tilde\pi_{1*}\left({\rm ch}_2((p\times {\rm Id}_X)^*\mathcal{I})\tilde\pi_2^*\gamma \cap [\widetilde{\mathfrak{I}}\times X]\right).$$

Thus by (3.10), to prove the lemma, it suffices to show that

$$\tilde\pi_{1*}\left({\rm ch}_2((p\times {\rm Id}_X)^*\mathcal{I})\tilde\pi_2^*\gamma \cap [\widetilde{\mathfrak{I}}\times X]\right) = \begin{cases} 0 & \text{if } \ell = 0 \text{ or } 1; \\ -\int_\beta \gamma \cdot [\widetilde{\mathfrak{I}}] & \text{if } \ell = 2. \end{cases} \tag{3.11}$$

Let $\mathcal{Z} \subset \mathfrak{I}\times X$ be the universal closed subscheme. Set-theoretically,

$$\mathcal{Z} = \{(I_Z, x) \in \mathfrak{I}\times X \,|\, x \in {\rm Supp}(Z)\}.$$

Let $\widetilde{\mathcal{Z}} = (p \times \mathrm{Id}_X)^{-1}\mathcal{Z}$. Then, $\mathcal{I} = I_{\mathcal{Z}}$, $(p \times \mathrm{Id}_X)^*\mathcal{I} = (p \times \mathrm{Id}_X)^* I_{\mathcal{Z}} = I_{\widetilde{\mathcal{Z}}}$, and

$$\mathrm{ch}_2((p \times \mathrm{Id}_X)^*\mathcal{I}) = \mathrm{ch}_2(I_{\widetilde{\mathcal{Z}}}) = -c_2(I_{\widetilde{\mathcal{Z}}}) = c_2(\mathcal{O}_{\widetilde{\mathcal{Z}}}). \tag{3.12}$$

If $\beta = 0$, then $\mathcal{Z}$ is of codimension-3 in $\mathfrak{I} \times X$, and $\widetilde{\mathcal{Z}}$ is of codimension-3 in $\widetilde{\mathfrak{I}} \times X$ as well. By (3.12), $\mathrm{ch}_2((p \times \mathrm{Id}_X)^*\mathcal{I}) = 0$. Therefore, (3.11) holds.

Next, we assume $\beta \neq 0$. Then, $\mathcal{Z}$ is of codimension-2 in $\mathfrak{I} \times X$, and $\widetilde{\mathcal{Z}}$ is of codimension-2 in $\widetilde{\mathfrak{I}} \times X$. By (3.12), $\mathrm{ch}_2((p \times \mathrm{Id}_X)^*\mathcal{I}) = -[\widetilde{\mathcal{Z}}]$. So

$$\tilde{\pi}_{1*}\Big(\mathrm{ch}_2((p \times \mathrm{Id}_X)^*\mathcal{I})\tilde{\pi}_2^*\gamma \cap [\widetilde{\mathfrak{I}} \times X]\Big) = -\tilde{\pi}_{1*}\big([\widetilde{\mathcal{Z}}] \cdot \tilde{\pi}_2^*\gamma\big). \tag{3.13}$$

When $\ell = 0$ or 1, we get $\tilde{\pi}_{1*}\big([\widetilde{\mathcal{Z}}] \cdot \tilde{\pi}_2^*\gamma\big) = 0$ by degree reason. Hence (3.11) holds.

We are left with the case $\ell = 2$. In this case, $\tilde{\pi}_{1*}\big([\widetilde{\mathcal{Z}}] \cdot \tilde{\pi}_2^*\gamma\big)$ is a multiple of $[\widetilde{\mathfrak{I}}]$. Let m be the multiplicity, and $\tilde{w} \in \widetilde{\mathfrak{I}}$ be a point. Then, we have

$$m = \deg\ \big([\widetilde{\mathcal{Z}}] \cdot \tilde{\pi}_2^*\gamma\big)|_{\{\tilde{w}\} \times X} = \int_\beta \gamma.$$

Therefore, we conclude from (3.13) that (3.11) holds when $\ell = 2$.

(ii) Follows from the proof of (i) by taking $\widetilde{\mathfrak{I}} = \mathfrak{I}$ and $p = \mathrm{Id}_{\mathfrak{I}}$. □

Lemma 3.2. (i) *Let $\beta \in H_2(X;\mathbb{Z})$. Then, we have*

$$\mathfrak{ch}_3(1_X)|_{H_*^{\mathrm{alg}}(\mathfrak{I}_n(X,\beta))} = -\left(n + \int_\beta K_X\right) \cdot \mathrm{Id}.$$

(ii) *If the moduli space $\mathfrak{I}_n(X,\beta)$ is smooth, then*

$$\mathfrak{ch}_3(1_X) = -\left(n + \int_\beta K_X\right) \cdot \mathrm{Id}. \tag{3.14}$$

Proof. Note that (i) follows from the proof of (ii) and the similar trick of using a nonsingular envelope as in the proof of Lemma 3.1 (i). To prove (ii), we adopt the notation in (3.4). Using the projection formula, we get

$$\mathfrak{ch}_3(1_X)(\xi) = \pi_{1*}\big(\mathrm{ch}_3(\mathcal{I}) \cap \pi_1^*\xi\big) = \pi_{1*}\mathrm{ch}_3(\mathcal{I}) \cdot \xi \tag{3.15}$$

since our moduli space $\mathfrak{I}_n(X,\beta)$ is smooth. Note that $\pi_{1*}\mathrm{ch}_3(\mathcal{I})$ is a multiple of the fundamental cycle of $\mathfrak{I}_n(X,\beta)$. Let m be the multiplicity. Then,

$$m = \deg\ \mathrm{ch}_3(\mathcal{I})|_{[I_Z] \times X} = \deg\ \mathrm{ch}_3(I_Z) = -\deg\ \mathrm{ch}_3(\mathcal{O}_Z) = -\frac{1}{2}\deg\ c_3(\mathcal{O}_Z)$$

where $[I_Z]$ denotes a point in $\mathfrak{I}_n(X,\beta)$. Since $c_1(\mathcal{O}_Z) = 0$ and $c_2(\mathcal{O}_Z) = -[Z] = -\beta$, we see from (3.1) and the Hirzebruch-Riemann-Roch Theorem that

$$m = -\frac{1}{2}\deg\ c_3(\mathcal{O}_Z) = -\left(n + \int_\beta K_X\right).$$

Now combining this with (3.15), we immediately obtain formula (3.14). □

Remark 3.3. Let $\beta \in H_2(X;\mathbb{Z})$ and $\gamma \in H^\ell(X;\mathbb{Q})$. We expect that both Lemma 3.1 and Lemma 3.2 can be sharpened, i.e., we expect in general that

$$\mathfrak{ch}_2(\gamma) = \begin{cases} 0 & \text{if } \ell = 0 \text{ or } 1; \\ -\int_\beta \gamma \cdot \mathrm{Id} & \text{if } \ell = 2; \end{cases}$$

$$\mathfrak{ch}_3(1_X) = -\left(n + \int_\beta K_X\right) \cdot \mathrm{Id}.$$

3.2. **The computations.**

In the rest of this section, we adopt the notation in Notation 2.2. We begin with the case when $n = 0$ and $\beta = d\beta_0$ with $d \geq 0$. Note that

$$\mathfrak{I}_0(X, d\beta_0) \cong S^{[d]}. \tag{3.16}$$

However, the expected dimension of $\mathfrak{I}_0(X, d\beta_0)$ is zero by (3.2).

Lemma 3.4. (i) *The obstruction bundle over the moduli space $\mathfrak{I}_0(X, d\beta_0) \cong S^{[d]}$ is isomorphic to the tangent bundle $T_{S^{[d]}}$ of the Hilbert scheme $S^{[d]}$.*

(ii) *The Donaldson-Thomas invariant $\langle\rangle_{0,d\beta_0}$ is equal to $\chi\big(S^{[d]}\big)$.*

Proof. It is clear that (ii) follows from (i). To prove (i), let

$$\psi = \mathrm{Id}_{S^{[d]}} \times f : S^{[d]} \times X \to S^{[d]} \times S$$

and $\phi : S^{[d]} \times S \to S^{[d]}$ be the projections. Let $\pi = \phi \circ \psi : S^{[d]} \times X \to S^{[d]}$. Let $\mathcal{J}$ be the universal ideal sheaf over $S^{[d]} \times S$. Then the universal ideal sheaf over

$$\mathfrak{I}_0(X, d\beta_0) \times X \cong S^{[d]} \times X$$

is $\mathcal{I} = \psi^*\mathcal{J}$. The Zariski tangent bundle and obstruction bundle over the moduli space $\mathfrak{I}_0(X, d\beta_0) \cong S^{[d]}$ are given by the rank-$2d$ bundles

$$\mathcal{E}xt^1_\pi(\psi^*\mathcal{J}, \psi^*\mathcal{J})_0, \ \mathcal{E}xt^2_\pi(\psi^*\mathcal{J}, \psi^*\mathcal{J})_0$$

respectively (see, for instance, the Theorem 3.28 in [Tho] for the obstruction bundle). Here $\mathcal{E}xt^*_\pi$ denotes the right derived functors of $\mathcal{H}om_\pi = \pi_*\mathcal{H}om$. We claim

$$\mathcal{E}xt^1_\pi(\psi^*\mathcal{J}, \psi^*\mathcal{J})_0 \cong \mathcal{E}xt^2_\pi(\psi^*\mathcal{J}, \psi^*\mathcal{J})_0. \tag{3.17}$$

In the following, we will prove the local version of (3.17), i.e., for every point $I_{f^*\xi} \in \mathfrak{I}_0(X, d\beta_0)$ with $\xi \in S^{[d]}$, we show that there exists a canonical isomorphism:

$$Ext^1(I_{f^*\xi}, I_{f^*\xi})_0 \cong Ext^2(I_{f^*\xi}, I_{f^*\xi})_0. \tag{3.18}$$

The argument for the global version (3.17) follows from that for the local version (3.18) and the isomorphisms via relative duality (see the Proposition 8.14 in [LeP]):

$$\begin{aligned} \mathcal{E}xt^2_\pi(\psi^*\mathcal{J}, \psi^*\mathcal{J})_0 &\cong \mathcal{E}xt^1_\pi(\psi^*\mathcal{J}, \psi^*\mathcal{J} \otimes \tilde{\rho}^*K_S)_0^\vee \\ \mathcal{E}xt^1_\phi(\mathcal{J}, \mathcal{J})_0 &\cong \mathcal{E}xt^1_\phi(\mathcal{J}, \mathcal{J} \otimes \rho^*K_S)_0^\vee \end{aligned}$$

where $\tilde{\rho} : S^{[d]} \times X = S^{[d]} \times S \times E \to S$ and $\rho : S^{[d]} \times S \to S$ are the projections.

Here is an outline for (3.18). We apply the Serre duality twice: once on X with

$$\begin{aligned} Ext^2(I_{f^*\xi}, I_{f^*\xi})_0 &\cong Ext^1(I_{f^*\xi}, I_{f^*\xi} \otimes K_X)_0^\vee \\ &\cong Ext^1(I_{f^*\xi}, I_{f^*\xi} \otimes f^*K_S)_0^\vee, \end{aligned}$$

and the other on S with $Ext^1(I_\xi, I_\xi)_0 \cong Ext^1(I_\xi, I_\xi \otimes K_S)_0^{\mathrm{v}}$. Note from (3.16) that

$$Ext^1(I_{f^*\xi}, I_{f^*\xi})_0 \cong Ext^1(I_\xi, I_\xi)_0. \tag{3.19}$$

The main part of our argument is to prove that there is a natural isomorphism:

$$Ext^1(I_{f^*\xi}, I_{f^*\xi} \otimes f^*K_S)_0 \cong Ext^1(I_\xi, I_\xi \otimes K_S)_0.$$

For simplicity, we assume that $\mathrm{Supp}(\xi) = \{s\} \subset S$. Note that the vector spaces $Ext^1(I_\xi, I_\xi)_0, Ext^1(I_{f^*\xi}, I_{f^*\xi})_0, Ext^2(I_{f^*\xi}, I_{f^*\xi})_0$ all have dimension $2d$.

Applying the local-to-global spectral sequence to $Ext^1(I_\xi, I_\xi)$, we obtain

$$0 \to H^1(S, \mathcal{O}_S) \to Ext^1(I_\xi, I_\xi) \to H^0\big(S, \mathcal{E}xt^1(I_\xi, I_\xi)\big) \to H^2(S, \mathcal{O}_S).$$

It follows that we have an exact sequence

$$0 \to Ext^1(I_\xi, I_\xi)_0 \to H^0\big(S, \mathcal{E}xt^1(I_\xi, I_\xi)\big) \to H^2(S, \mathcal{O}_S). \tag{3.20}$$

Since the second term can be computed locally, by taking $S = \mathbb{P}^2$, we see that

$$h^0\big(S, \mathcal{E}xt^1(I_\xi, I_\xi)\big) = 2d$$

for an arbitrary surface S. So we conclude from (3.20) that

$$Ext^1(I_\xi, I_\xi)_0 \cong H^0\big(S, \mathcal{E}xt^1(I_\xi, I_\xi)\big) \tag{3.21}$$

since $\dim Ext^1(I_\xi, I_\xi)_0 = 2d$. Similarly, we have canonical isomorphisms:

$$\begin{aligned} Ext^1(I_{f^*\xi}, I_{f^*\xi})_0 &\cong H^0\big(X, \mathcal{E}xt^1(I_{f^*\xi}, I_{f^*\xi})\big) \\ &\cong H^0\big(S, f_*\mathcal{E}xt^1(I_{f^*\xi}, I_{f^*\xi})\big). \end{aligned} \tag{3.22}$$

As in (3.20), we have an injection

$$0 \to Ext^1(I_\xi, I_\xi \otimes K_S)_0 \to H^0\big(S, \mathcal{E}xt^1(I_\xi, I_\xi \otimes K_S)\big).$$

Note that $H^0\big(S, \mathcal{E}xt^1(I_\xi, I_\xi \otimes K_S)\big) \cong H^0\big(S, \mathcal{E}xt^1(I_\xi, I_\xi)\big) \otimes_{\mathbb{C}} K_S|_s$ since $\mathcal{E}xt^1(I_\xi, I_\xi)$ is supported at $\mathrm{Supp}(\xi) = \{s\}$, where $K_S|_s$ is the fiber of K_S at $s \in S$. So we get

$$0 \to Ext^1(I_\xi, I_\xi \otimes K_S)_0 \to H^0\big(S, \mathcal{E}xt^1(I_\xi, I_\xi)\big) \otimes_{\mathbb{C}} K_S|_s.$$

By (3.21) and the Serre duality, $Ext^1(I_\xi, I_\xi \otimes K_S)_0$ and $H^0\big(S, \mathcal{E}xt^1(I_\xi, I_\xi)\big)$ have the same dimension. Hence, we get an isomorphism

$$Ext^1(I_\xi, I_\xi \otimes K_S)_0 \cong H^0\big(S, \mathcal{E}xt^1(I_\xi, I_\xi)\big) \otimes_{\mathbb{C}} K_S|_s. \tag{3.23}$$

Again as in (3.20), we have another injection:

$$0 \to Ext^1(I_{f^*\xi}, I_{f^*\xi} \otimes f^*K_S)_0 \to H^0\big(X, \mathcal{E}xt^1(I_{f^*\xi}, I_{f^*\xi} \otimes f^*K_S)\big).$$

By the Serre duality, $Ext^1(I_{f^*\xi}, I_{f^*\xi} \otimes f^*K_S)_0 \cong Ext^2(I_{f^*\xi}, I_{f^*\xi})_0^{\mathrm{v}}$. Also,

$$\begin{aligned} H^0\big(X, \mathcal{E}xt^1(I_{f^*\xi}, I_{f^*\xi} \otimes f^*K_S)\big) &\cong H^0\big(S, f_*\mathcal{E}xt^1(I_{f^*\xi}, I_{f^*\xi} \otimes f^*K_S)\big) \\ &\cong H^0\big(S, f_*\mathcal{E}xt^1(I_{f^*\xi}, I_{f^*\xi}) \otimes K_S\big) \\ &\cong H^0\big(S, f_*\mathcal{E}xt^1(I_{f^*\xi}, I_{f^*\xi})\big) \otimes_{\mathbb{C}} K_S|_s \end{aligned}$$

since $f_*\mathcal{E}xt^1(I_{f^*\xi}, I_{f^*\xi})$ is supported on $\mathrm{Supp}(\xi) = \{s\}$. Therefore, we obtain

$$0 \to Ext^2(I_{f^*\xi}, I_{f^*\xi})_0^{\mathrm{v}} \to H^0\big(S, f_*\mathcal{E}xt^1(I_{f^*\xi}, I_{f^*\xi})\big) \otimes_{\mathbb{C}} K_S|_s. \tag{3.24}$$

Since $Ext^2(I_{f^*\xi}, I_{f^*\xi})_0$ and $Ext^1(I_{f^*\xi}, I_{f^*\xi})_0$ have the same dimension, we obtain

$$\begin{aligned} Ext^2(I_{f^*\xi}, I_{f^*\xi})_0^{\vee} &\cong H^0\big(S, f_*\mathcal{E}xt^1(I_{f^*\xi}, I_{f^*\xi})\big) \otimes_{\mathbb{C}} K_S|_s \\ &\cong Ext^1(I_{f^*\xi}, I_{f^*\xi})_0 \otimes_{\mathbb{C}} K_S|_s \end{aligned}$$

from (3.22) and (3.24). Combining this with (3.19) and (3.21), we get

$$Ext^2(I_{f^*\xi}, I_{f^*\xi})_0^{\vee} \cong H^0\big(S, \mathcal{E}xt^1(I_\xi, I_\xi)\big) \otimes_{\mathbb{C}} K_S|_s. \tag{3.25}$$

In view of (3.23), the Serre duality and (3.19), we conclude that

$$Ext^2(I_{f^*\xi}, I_{f^*\xi})_0 \cong Ext^1(I_\xi, I_\xi \otimes K_S)_0^{\vee} \cong Ext^1(I_\xi, I_\xi)_0 \cong Ext^1(f^*I_\xi, f^*I_\xi)_0.$$

This completes the proof of the isomorphism (3.18). □

Next, we consider the case when either $n \neq 0$ or $\beta \neq d\beta_0$ with $d \geq 0$. We further assume that the moduli space $\mathfrak{I}_n(X, \beta)$ is nonempty. For simplicity, put

$$\mathfrak{I} = \mathfrak{I}_n(X, \beta).$$

Let $\mathcal{I}$ be the universal ideal sheaf over $\mathfrak{I} \times X$. Denote the trace-free part of the element $R\mathcal{H}om(\mathcal{I}, \mathcal{I})$ in the derived category $\mathfrak{D}_{\text{coh}}(\mathfrak{I} \times X)$ by

$$R\mathcal{H}om(\mathcal{I}, \mathcal{I})_0.$$

Let $\pi : \mathfrak{I} \times X \to \mathfrak{I}$ be the projection. By [Tho], the virtual fundamental class $[\mathfrak{I}]^{\text{vir}}$ is defined via the following element in the derived category $\mathfrak{D}_{\text{coh}}(\mathfrak{I})$:

$$\mathcal{E} = R\pi_*\big(R\mathcal{H}om(\mathcal{I}, \mathcal{I})_0\big). \tag{3.26}$$

Let $p \in E$, and consider the sheaf $\big(\text{Id}_{\mathfrak{I}} \times \phi_p \times \text{Id}_S\big)^*\mathcal{I}$ over

$$\mathfrak{I} \times X = \mathfrak{I} \times E \times S.$$

We see from (2.18) that $\big(\text{Id}_{\mathfrak{I}} \times \phi_p \times \text{Id}_S\big)^*\mathcal{I}$ is a flat family of ideal sheaves whose corresponding 1-dimensional closed subschemes satisfy (3.1). By the universal property of the moduli space $\mathfrak{I}$, there is an automorphism

$$\Phi_p : \mathfrak{I} \to \mathfrak{I} \tag{3.27}$$

such that $(\Phi_p \times \text{Id}_X)^*\mathcal{I} = \big(\text{Id}_{\mathfrak{I}} \times \phi_p \times \text{Id}_S\big)^*\mathcal{I} \cong \mathcal{I}$. In particular, E acts on $\mathfrak{I}$.

Lemma 3.5. *Let $n \neq 0$ or $\beta \neq d\beta_0$ with $d \geq 0$. Then,*

$$\langle \tilde{\tau}_{k_1}(\gamma_1) \cdots \tilde{\tau}_{k_r}(\gamma_r) \rangle_{n,\beta} = 0 \tag{3.28}$$

whenever $\gamma_1, \ldots, \gamma_r \in f^*H^*(S; \mathbb{Q}) \subset H^*(X; \mathbb{Q})$.

Proof. The proof is similar to that of Lemma 2.5. Assume that the moduli space

$$\mathfrak{I} = \mathfrak{I}_n(X, \beta)$$

is nonempty. If $\beta \neq d\beta_0$ with $d \geq 0$, then the algebraic group E acts on $\mathfrak{I}$ with finite stabilizers. If $\beta = d\beta_0$ with $d \geq 0$ and if $I_Z \in \mathfrak{I}$, then Z consists of a curve $f^*(\xi)$ for some $\xi \in S^{[d]}$ and of some (possibly embedded) points of length $n \neq 0$. So again E acts on the moduli space $\mathfrak{I}$ with finite stabilizers.

As in the proof of Lemma 2.5, there exists some number N such that if G is a cyclic subgroup of E of prime order $p > N$, then G acts freely on $\mathfrak{I}$. Fix such

cyclic subgroups G of E. Since the complex $R\pi_*\big(R\mathcal{H}om(\mathcal{I},\mathcal{I})_0\big)$ from (3.26) is equivariant for the action of any algebraic automorphism group of X, the cycle $[\mathfrak{I}]^{\rm vir}$ defines an element of the equivariant Borel-Moore homology group $H_*^G(\mathfrak{I})$. For $1 \le i \le r$, choose a positive integer m_i such that the multiple $m_i\gamma_i$ defines an element of $H_G^*(X)$. It follows from (3.4) and (3.5) that the cycle

$$m_1 \cdots m_r \ \langle \tilde\tau_{k_1}(\gamma_1) \cdots \tilde\tau_{k_r}(\gamma_r)\rangle_{n,\beta}$$

defines an element in the degree-0 Borel-Moore homology $H_0^G(\mathfrak{I})$. Again as in the proof of Lemma 2.5, we conclude that $\langle \tilde\tau_{k_1}(\gamma_1) \cdots \tilde\tau_{k_r}(\gamma_r)\rangle_{n,\beta} = 0$. □

Proposition 3.6. *Let $\beta \in H_2(X;\mathbb{Z})\backslash\{0\}$. Assume $\int_\beta K_X = \int_\beta f^*K_S = 0$. Then,*

$$\mathbf{Z}'_{\rm DT}\left(X;q\Big|\prod_{i=1}^r \tilde\tau_0(\gamma_i)\right)_\beta = \begin{cases} \prod_{i=1}^r \int_\beta \gamma_i \cdot \chi\big(S^{[d]}\big) & \textit{if } |\gamma_i| = 2 \textit{ for every } i \textit{ and } \beta = d\beta_0 \textit{ for some } d \ge 1; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. First of all, since $\chi(X) = 0$, we see from (3.8) and (3.6) that

$$\mathbf{Z}'_{\rm DT}\left(X;q\Big|\prod_{i=1}^r \tilde\tau_{k_i}(\gamma_i)\right)_\beta = \sum_{n\in\mathbb{Z}} \langle \tilde\tau_{k_1}(\gamma_1) \cdots \tilde\tau_{k_r}(\gamma_r)\rangle_{n,\beta}\ q^n. \tag{3.29}$$

Next, in view of (3.2) and the condition on degrees, we have

$$\sum_{i=1}^r |\gamma_i| = 2r, \qquad |\gamma_r| \le 2.$$

Therefore, we conclude from (3.5) and Lemma 3.1 (i) that

$$\langle \tilde\tau_0(\gamma_1) \cdots \tilde\tau_0(\gamma_r)\rangle_{n,\beta} = \begin{cases} \prod_{i=1}^r \int_\beta \gamma_i \cdot \langle\rangle_{n,\beta} & \text{if } |\gamma_i| = 2 \text{ for every } i; \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 3.4 (ii) and Lemma 3.5, we obtain

$$\langle \tilde\tau_0(\gamma_1) \cdots \tilde\tau_0(\gamma_r)\rangle_{n,\beta} = \begin{cases} \prod_{i=1}^r \int_\beta \gamma_i \cdot \chi\big(S^{[d]}\big) & \text{if } |\gamma_i| = 2 \text{ for every } i, n = 0, \beta = d\beta_0; \\ 0 & \text{otherwise.} \end{cases}$$

Now the proposition follows immediately from (3.29). □

Proposition 3.7. *Let X be from Notation 2.2 and $\beta \in H_2(X;\mathbb{Z})\backslash\{0\}$. Assume that $\gamma_1, \ldots, \gamma_r \in f^*H^*(S;\mathbb{Q}) \subset H^*(X;\mathbb{Q})$. Then,*

$$\mathbf{Z}'_{\rm DT}\left(X;q\Big|\prod_{i=1}^r \tilde\tau_{k_i}(\gamma_i)\right)_\beta = \begin{cases} \chi\big(S^{[d]}\big) & \textit{if } r = 0 \textit{ and } \beta = d\beta_0 \textit{ with } d \ge 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $\beta \neq d\beta_0$, then the proposition follows from (3.29) and Lemma 3.5. In the rest of the proof, we let $\beta = d\beta_0$ with $d \geq 1$. By (3.29) and Lemma 3.5 again,

$$\mathbf{Z}'_{\mathrm{DT}}\left(X; q| \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i)\right)_{d\beta_0} = \langle \tilde{\tau}_{k_1}(\gamma_1) \cdots \tilde{\tau}_{k_r}(\gamma_r)\rangle_{0,d\beta_0}.$$

Thus we see from Lemma 3.4 (ii) that the proposition holds if $r = 0$.

To prove our proposition, it remains to verify that if $r \geq 1$, then

$$\langle \tilde{\tau}_{k_1}(\gamma_1) \cdots \tilde{\tau}_{k_r}(\gamma_r)\rangle_{0,d\beta_0} = 0. \tag{3.30}$$

Since the expected dimension of $\mathfrak{I}_0(X, d\beta_0)$ is zero, (3.30) holds unless

$$\sum_{i=1}^{r}(2k_i - 2 + |\gamma_i|) = 0, \qquad (2k_r - 2 + |\gamma_r|) \leq 0. \tag{3.31}$$

W.l.o.g., we may assume that $k_{\tilde{r}+1} = k_{\tilde{r}+2} = \ldots = k_r = 0$ and

$$k_1, \ldots, k_{\tilde{r}-1}, k_{\tilde{r}} \geq 1 \tag{3.32}$$

for some $\tilde{r}$ with $0 \leq \tilde{r} \leq r$. Then we see from (3.5), Lemma 3.1 (i) and (3.31) that (3.30) holds unless $\tilde{r} = r$, $k_1 = \ldots = k_r = 1$, and $|\gamma_1| = \ldots = |\gamma_r| = 0$. When

$$k_1 = \ldots = k_r = 1, \quad |\gamma_1| = \ldots = |\gamma_r| = 0,$$

(3.30) follows from Lemma 3.2 (ii) since the moduli space $\mathfrak{I}_0(X, d\beta_0)$ is smooth. $\square$

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: `edidin@math.missouri.edu`

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: `zq@math.missouri.edu`